15.1 Counting Problems and Permutaions

Period 7

Theorem 15-1

The Fundamental Counting Principle

In a compound event, the first event occurs in n_1 ways, the second event occurs in n_2 ways, and so on, and the k^{th} event occurs in n_k ways. The total number of ways the compound event may occur is

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$$

Ex.: Margaret has three pairs of shoes, four skirts, and five blouses. How many different outfits can she assemble?

3 ways to pick shoes
4 ways to pick skirts
5 ways for blouses
$3 \cdot 4 \cdot 5 = 60$ ways for an outfit

Permutations

A permutation is a set of n objects is an **ordered** arrangement of the objects.
Can not be repeated

Theorem 15-2

${}_nP_n \rightarrow$ the number of permutations of n objects taken n at a time

$${}_nP_n = n(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$$

Ex.: In how many different ways can Anne, Bob, Carmine and Della be arranged in a line?

1st Position - 4
2nd Position - 3
3rd Position - 2
4th Position - 1

$4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways

$4! = 4 \cdot 3 \cdot 2 \cdot 1$

! Factorial

${}_4P_4$

$$_nP_n = n(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1$$

Ex.: $_8P_8 =$ $8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 40{,}320$

Ex.: In how many different ways can six circus elephants be arranged in a line?

$_6P_6$ $\quad 6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$

Factorial Notation
$n!$ (n factorial) = $n(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1$
$0! = 1$ by definition
$\therefore \quad {}_nP_n = n!$

Ex.: $5! =$ $5\cdot4\cdot3\cdot2\cdot1 = 120$

$7! =$ $7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 5040$

$1! =$ 1

$$\boxed{n! = n(n-1)!}$$

$$8! = 8\cdot7!$$
$$= 8\cdot(7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)$$

Ex.: Rewrite $7!$ with a factor of $5!$ $\quad 7\cdot6\cdot5!$

$$\frac{7!}{5!} = \frac{7\cdot6\cdot\cancel{5!}}{\cancel{5!}} = 42$$

Permutations of *n* Objects Taken *r* at a Time
(*without replacement)

Theorem 15-4

The number of permutations of a set of n objects taken r at a time, denoted ${}_nP_r$, is given by

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Ex.: Compute ${}_4P_2 =$ $\frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4\cdot 3\cdot \cancel{2!}}{\cancel{2!}} = 12$

Total set   # you choose

${}_7P_3 =$ $\frac{7!}{(7-3)!}$ $\frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}} = 210$

In an Olympic event with five competitors, how many different orderings are there for the gold medal, silver medal and bronze medal winners?

${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot 4\cdot 3\cdot \cancel{2!}}{\cancel{2!}} = 60$

Per 4

A teacher wants to write an ordered 6-question test from a pool of 10 questions. How many different forms of the test can she write?

${}_{10}P_6 = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}}$

151,200

15.2 Permutations for Special Counts

Theorem 15-5

The number of orderings of n objects taken r at a time, with repetition, is n^r.

How many different ways can a sequence of 4 cards be dealt from a full deck of 52 cards:

a) without replacement? $_{52}P_4$ $\frac{52!}{(52-4)!} = \frac{52!}{48!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot \cancel{48!}}{\cancel{48!}}$

6,497,400

b) with replacement? $52 \cdot 52 \cdot 52 \cdot 52 = 52^4 = n^4 = n^r$

7,311,616

Permutations with Identical Objects

Theorem 15-6

The number of permutations, P, of n objects taken n at a time, with r objects alike, s of another kind alike, t of another kind alike, is

$$P = \frac{n!}{r!s!t!}$$

Find the number of permutations of the letters of the word *BANANAS*.

7 letters

A - 3
N - 2

$\frac{7!}{3! \cdot 2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}$

A N

$\frac{7 \cdot 6 \cdot 5 \cdot \cancel{4}\, 2}{\cancel{2} \cdot 1} =$

$\boxed{420}$

15.3 Combinations

combinations – the number of ways that we can select elements from a set
(*without regard to their order)

How many combinations are there of {A, B, C, D} taken

a) four at a time? 1

b) three at a time?

~~ABC~~ BCD
~~ABD~~ BAD
~~ADB~~ BDC

ABC
ABD
ACD
BCD

4

$\binom{n}{r}$ → "n choose r" → Denotes the number of ways we can select r elements from a set containing n elements.

Theorem 15-8
The number of combinations of a set of ***n*** objects taken ***r*** at a time is

$${}_nC_r \qquad \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Ex.: Simplify.

$$\binom{5}{2} \quad {}_5C_2 = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot 2!} = 10$$

Ex.: For a study, 4 people are chosen at random from a group of 10 people. In how Many ways can this be done?

$${}_{10}C_4 = \frac{10!}{(10-4)!\cdot 4!} = \frac{10!}{6!\cdot 4!} \quad \frac{10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}\cdot 4!}$$

$$\frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 210$$

Ex.: How many committees can be formed from a set of 5 governors and 7 senators if each committee contains 3 governors and 4 senators.

Governers

$${}_5C_3 = \frac{5!}{(5-3)!\cdot 3!} = \frac{5!}{2!\cdot 3!} = \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot 2!} = 10$$

Senaters

$${}_7C_4 = \frac{7!}{(7-4)!\,4!} = \frac{7!}{3!\cdot 4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{3!\cdot \cancel{4!}} = \frac{7\cdot \cancel{6}\cdot 5}{\cancel{3\cdot 2}\cdot 1} = 35$$

6

$$10\cdot 35 = 350$$